Knot Theory Lectures

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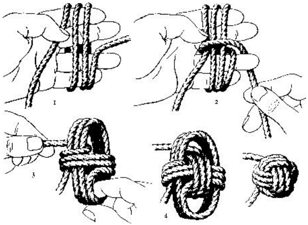
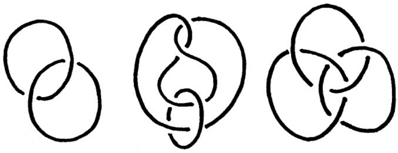
Knot theory

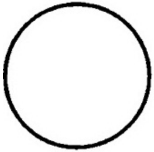
Knot theory is the area of topology that studies mathematical knots and links. A knot is an entwined circle, or precise mathematical language, a knot is an embedded circle in 3-dimensional Euclidean space $\mathbf{R}^3$, *i.e.* One dimensional non-intersecting curve in $\mathbf{R}^3$.



A *link* or a *knot* in $\mathbf{S}^3$ is a smooth embedding of a disjoint family of circles (link) or a single circle (knot), *i.e.* it is a collection of disjoint smooth simple closed curves, which is a 1-dimensional closed submanifold of $\mathbf{S}^3$.



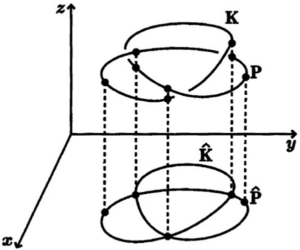
A link (knot) is said to be trivial if it is equivalent to a circles (circle).



The relation between equivalence and ambient isotopy is the following. Given a choice of the orientation on the 3-sphere $\mathbf{S^3}$, if the homeomorphism of $\mathbf{S^3}$ that gives the equivalence between L_1 and L_2 is orientation preserving, then there is a continuous family of homeomorphisms of $\mathbf{S^3}$ beginning from the identity and ending with a homeomorphism taking L_1 to L_2 which is an ambient isotopy. Thus, two links L_1 and L_2 are ambient isotopic if there exists an orientation preserving homeomorphisms φ of $\mathbf{S^3}$ with $L_1 = \varphi(L_2)$.

A knot is said to be tame if it is isotopic to a polygonal knot. Non-tame knot exist and are called wild. The set of tame knot types is countable. A knot is called smooth if it is a smooth submanifold of $\mathbf{S}^3$.
Let K be a tame knot type. One can project K onto a plane in such a way that the image is a nodal curve. By drawing the nodal points as crossings that remember the 3-dimensional positions of the two crossing strands of the knot, one obtains a picture called a *knot diagram D* of K. One can define the link diagrams in the same way.

Let us denote by P the map projects point $P(x, y, z)$ in $\mathbb{R}^3$ onto the point $\hat{P}(x, y, 0)$ in the xy-plane



We shall say that $P(K) = \hat{K}$ is the projection of K. If K has an orientation assigned, then in a natural way $\hat{K}$ inherits its orientation from the orientation of K.

$\hat{K}$ is not a simple closed curve lying on the plane, since $\hat{K}$ possesses several points of intersection. By performing several elementary knots moves on K, we can impose the following conditions:-

1. $\hat{K}$ has at most finite number of points of intersection.
2. If Q is a point of intersection of $\hat{K}$, then the inverse image $P^{-1} \cap K$, of Q in K has exactly two points, that is, Q is a double point of $\hat{K}$.
3. A vertex of K is never mapped onto a double point of $\hat{K}$.

A projection $\hat{K}$ that satisfies the above conditions is said to be a regular projection.

Definition

We say that two knots K_1 and K_2 are equivalent, if there exist an orientation-preserving homeomorphism of $\mathbf{R^3}$ that map K_1 to K_2.

Lemma

Let $f : \mathbf{R^3} \longrightarrow \mathbf{R^3}$ be a homoeomorphism map that map K_1 to K_2 where K_1, K_2 knots in $\mathbf{R^3}$. If f is orientation-preserving, then K_1 ambient isotopic to K_2

The 3-dimensional sphere $\mathbf{S^3}$ is often thought of as $\mathbb{R}^3$ with a point ∞ added at infinity. It will be more convenient to think of a knot lying in $\mathbf{S^3}$ rather that $\mathbb{R}^3$.

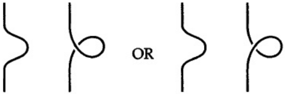
Let K be an embedded knot in $\mathbf{S^3}$. We define the knot complement as the complement of the knot in $\mathbf{S^3}$ *i.e.* the topological space $\mathbf{S^3} - K$. Let K_1 and K_2 be two ambient isotopic knots in $\mathbf{S^3}$, and let $f : \mathbf{S^3} \longrightarrow \mathbf{S^3}$ be an orientation preserving homeomorphism of $\mathbf{S^3}$ with $f(K_1) = K_2$. The restriction $f|(\mathbf{S^3} - K_1) \longrightarrow (\mathbf{S^3} - K_2)$ is also a homeomorphism. Thus, two ambient isotopic knots have homeomorphic knot complements.

Theorem

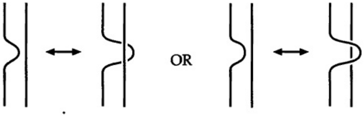
If two knots K_1 and K_2 that lie in $\mathbf{S^3}$ are equivalent then their complements $\mathbf{S^3} - K_1$ and $\mathbf{S^3} - K_2$ are homeomorphic.

If we have different regular diagram of the same knot, we can get one to look like the other using finite simple types of moves called Reidemeister Moves. A Reidemeister Move is one of three ways to change the regular diagram of the knot that will change the relation between the crossings.
The first Reidemeister move allows us to put in or take out a twist in the knot.
The second Reidemeister move allows us to either add two crossings or remove two crossings.
The third Reidemeister move allows us to slide a strand of the knot from one side of a crossing to other side of the crossing.
Each of these moves changes the regular diagram of the knot.

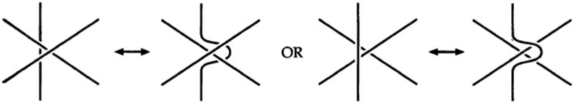
Two diagrams represent the same link or knot type if and only if we can get one from the other by finite sequence of Reidemeister moves



Type I Reidemeister move.



Type II Reidemeister move.



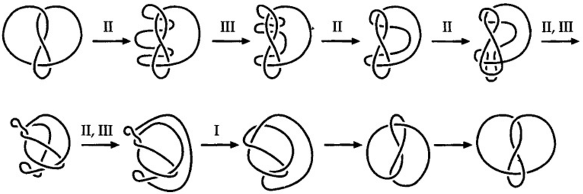
Type III Reidemeister move.

The Reidemeister Theorem give us the relation between the equivalence of knots in a space and their equivalence diagrams. From all these definitions we can define an equivalent class:-

$$[K] = \{K', K' \quad \text{knot in} \quad \mathbb{R}^3 | K \simeq K'\}$$

Each equivalent class of knots called knot type.

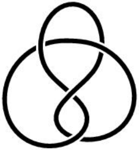
Reidemeister moves.



The figure-eight knot is equivalent to its mirror image.

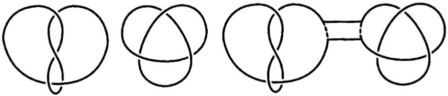
Chirality

Suppose that we have a knot in $\mathbb{R}^3$ representing some knot K. Its image by a reflection through a plane is naturally called mirror image of the knot. The reflection is a homeomorphism which reverses the orientation of $\mathbb{R}^3$. Let K be a knot type in $\mathbf{S^3}$. We can reflect its image through a plane to get a knot K^m called the mirror image of K. If K ambient isotopic to its mirror image K^m then K is called achiral and if they are not ambient isotopic then the knot is called chiral. For example, the Figure-8 knot is achiral.



Composition of Knots

If we have two knots in$\mathbb{R}^3$, then we can define a new knot by removing a small arc from each and then glue the four and points by two new arcs (need picture) We called the resulting knot the composition of the two knots K_1 and K_2 and denoted by $K_1 \# K_2$.



A knot or a link called reducible (composite knot) if it can be expressed as the *connected sum* of two non-trivial knots or links. Recall that, if we have two knots K_1 and K_2, then the connected sum of K_1 and K_2, denoted by $K_1 \# K_2$, is formed in the following way. Take a knot projection of K_1 and K_2, and put them next to each other. Select a small *arc* on each of the two knots K_1 and K_2. Delete a segment of arc from each, and connect the endpoints by adding two new arcs each connecting an endpoint on one of the two knots to an endpoint on the other. A knot is composite if it is a connected sum of two non-trivial knots.

The knots K_1 and K_2 are *factor knots* of $K_1 \# K_2$. The decomposition of knots into *prime* factors is unique up to the order of each summand in the connected sum (like the unique prime factorization of natural numbers). For example, the trefoil knot is a prime knot.

However, unlike the case of prime numbers, here there are two choices of how to connect the endpoints of the arcs in performing a connected sum. These yield the same result whenever one of the knots is invertible.

Theorem

The composition $K_1 \# K_2$ is unique if and only If one of the two knots K_1 or K_2 is invertible (i.e. it can be deformed by an ambient isotopy into the same knot with the reverse orientation).

Remark

1. The composition is independent of choice of the two arcs by Reidemeister moves *i.e.* if $K_1 \# K_2 = K$ on arcs (need pictures). In remak above we saw that the composition is unique without orientation but the orientation it is different.

The composition of two knots is commutative *i.e.*

$$K_1 \# K_2 \simeq K_2 \# K_1$$

with or without ordination.

we can shrink one knot and slide it along the second knot until it reaches to the other side.

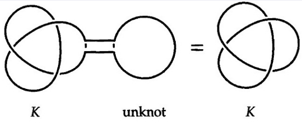
Remark

1. The composition operation is associative *i.e.*

$$K_1 \# (K2 \# K_3) \simeq (K_1 \# K_2) \# K_3$$

for $K_1, K_2,$ and K_3 knots in $\mathbb{R}^3$

2. The knot K_1 or K_2 is called factor knot of the knot K
3. $K_1 \# \bigcirc$ (trivial knot) $= K_1$ *i.e.* there is no composition and $\bigcirc$ behaves like identity



K#(unknot) is just K.

so we want the two knots K_1 and K_2 to be non-trivial knots.

Remark

1. If a knot is not the composition of any two non-trivial knots then we call it **prime knot**, for example the trefoil knot is a prime knot.
2. Any knot is either prime or is a composition of two non-trivial knots, these two knots either prime knots or one of them or both are composition of non-trivial knots and so on.
3. The trivial knot is not composition of any two knots to show that suppose that $\bigcirc = K_1 \# K_2$ where K_1, K_2 non-trivial prime knots (or at least one of them prime say K_1) then

$$K_1 = K_1 \# \bigcirc = K_1 \# K_1 \# K_2$$

so this means that K_1 is not prime which is a contradiction so we want the two knots K_1 and K_2 to be non-trivial knots.

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Definition

If K is oriented knot then the knot with opposite orientation denoted $-K$ is called reverse of K. (need picture)

One can ask if the set of all knots (oriented or not) is a group. The answer is no because we didn't have to every knot K an inverse element *i.e.* $\forall K$, where K is a knot, there is no knot K^{-1} such that $K \# K^{-1} \simeq \bigcirc$, therefore the set of all knots in $\mathbb{R}^3$ is a semigroup with identity for the composition operation $\#$.

Theorem

1. *Any knot can be decomposed into finite number of prime knots.*
2. *This decomposition excluding the order, is unique, that is say, suppose we can decompose K into two ways:-*

$$K_1, K_2, ..., K_m \quad \text{and} \quad K'_1, K'_2, ..., K'_n$$

then

- $m = n$
- $K_1 \simeq K'_1, K_2 \simeq K'_2, ..., K_m \simeq K'_m$

The Minimum Number of Crossing Points

A regular diagram D of a knot (or link) K has at most a finite number of crossing points denoted this number $c(D)$. which is not knot invariant to show that: the trivial knot has two regular diagrams D and D' which have a different number of crossing points.

Definition

The minimum number of crossing points of all regular diagrams of K denoted by $c(K)$ which is a knot invariant.

Theorem

$c(K) = \min_{D} c(D)$ is a knot invariant, where D is the set of all regular diagram of K.

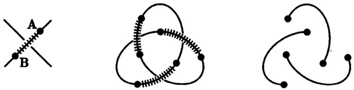
The crossing number of the trivial knot $\bigcirc$ is equal 0, $c(\bigcirc) = 0$
Notes:

1. There is no known method to determine $c(K)$
2. $c(K)$ has been completely determined in case of alternating knots (or links)
3. $c(K)$ has also been determined for some specific types of non-alternating knots.
4. There exist a conjecture about the composition of two knots as follows:
   Suppose that K_1 and K_2 are two arbitrary knots (or links) then
   $$c(K_1 \# K_2) = c(K_1) + c(K_2)$$
   In special case when K_1 and K_2 are alternating knots this conjecture is true.

If we have a regular diagram D of a knot (or link) then at each crossing point of D we can remove a small segment $\overline{AB}$ as the fig (need picture) that pass over the crossing point. The result of removing these segments is a collection of disconnected polygonal curves and since these segments pass over the segments on the plane, then these segments are called **Bridges**

The bridge number of a regular diagram D is not a knot invariant for a knot K. There exist knot have regular diagrams with different bridge numbers (by Reidemeister Moves).

If we consider all the regular diagrams for a given knot then the minimum bridge number of all these regular diagrams is an invariant for the knot K.



Theorem

For a knot (or link) $br(K) = \min_{D} br(D)$ *is an invariant for* K *where* D *is the set of all regular diagrams of* K*. This quantity is called the bridge number (or bridge index) of* K*.*

Actually no method has yet been found to allow us o determine $br(K)$ for an arbitrary knot K.

For composition of two knots K_1, K_2 we have this theorem.

Theorem

Suppose that K_1 *and* K_2 *are two arbitrary knots (or links) then*

$$br(K_1 \# K_2) = br(K_1) + br(K_2) - 1$$

Definition

The bridge number is the number of bridge for a given diagram D for a knot (or link) K and denoted by $br(D)$

Definition

Suppose D is a regular diagram of a knot (link) K. If we can divide D into $2n$ polygonal curves $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ *i.e.* $D = \alpha_1 \cup \alpha_2 \cup \ldots, \alpha_n \cup \beta_1 \cup \beta_2 \cup \ldots \cup \beta_n$ That satisfy the condition given below, then the bridge number of D $(br(D)$ is said to be at most n.

1. $\alpha_1, \alpha_2, \ldots, \alpha_n$ are mutually disjoint, simple polygonal curves
2. $\beta_1, \beta_2, \ldots, \beta_n$ are also mutually disjoint, simple curves.
3. At the crossing point of D $\alpha_1, \alpha_2, \ldots, \alpha_n$ are segment that pass over the crossing points, while $\beta_1, \beta_2, \ldots, \beta_n$ are segments that pass under the crossing points.

If $br(D) \leq n$ but $br(D) \nleq n-1$ then we define $br(D) = n$

If $br(K) = 1$ then K is the trivial knot.
The connected sum of n copies of a trefoil knot has the bridge number $n+1$, but in comparison to $c(K)$, $br(K)$ is usually quite small, but the two quantities are quite closely related as the following conjecture

Conjecture:
If K is a knot, then $c(K) \geq 3(br(K) - 1)$ where equality only holds when K is trivial knot, the trefoil knot or the composition of trefoil knots.

At one of the crossing points of a regular diagram D of a knot (or link) K exchange, locally the over-and under-crossing segments. What we obtain is a regular diagram of some other knot If we exchange the under-and over- crossing segments within the small circle, the subsequent regular diagram can readily be seen to be that of trivial knot. We can exchange a regular diagram D of an arbitrary knot (or link) to the regular diagram of trivial knot (or link) by exchange the over-and under- crossing segments at several crossing points of D.

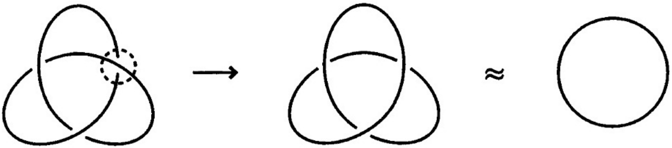
From above proposition the operation which exchange the over-and under- crossing segments at a crossing points is called unknotting operation.

Definition

Unknotting number is the minimum number of unknotting operations that are required to change the regular diagram D of a knot (or link) K into the regular diagram of the trivial knot (or link) and denoted by $u(D)$, but $u(D)$ is not an invariant of K.

Theorem

If K is a knot (or link), then $u(K) = \min_{D} u(D)$ is an invariant of K where D is the set of all regular diagrams of K. We say that $u(K)$ is the unknotting number of K.



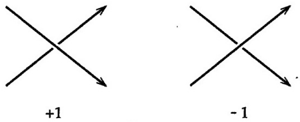
Remarks:

1. If K is a non trivial knot then $u(K) \geq 1$.
2. Determine $u(K)$ is a very hard problem, even for every specific types of knots. There are virtually no method to determine $u(K)$.

The linking number is an important invariant to oriented links.

Definition

A crossing point c of an oriented regular diagram we have two possible configurations see fig



Computing linking number.

In case (1) we assign $Sign(c) = +1$ to the crossing point and in case (2) we assign $Sign(c) = -1$.

If a crossing is of case (1), then rotating the under strand clockwise lines it up with the over strand so that their arrow match. Similarly, If a crossing is of the second type, then rotating the under strand counter clockwise lines it up with the over strand so that their arrows match. Suppose that D is an oriented regular diagram of a 2-components link $L = \{K_1, K_2\}$ and suppose also that the crossing points of D at which the projection of K_1 and K_2 intersect are $c_1, c_2, ..., c_m$ (we ignore the crossing points of the projection of K_1 and K_2 which are self-intersection of the knot component) then

$$\frac{1}{2}(Sign(c_1) + Sign(c_2) + ... + Sign(c_m))$$

is called the linking number of K_1 and K_2 and denoted by $lk(K_1, K_2)$

Theorem

The linking number $lk(K_1, K-2)$ is an invariant of L, i.e. If we consider another oriented diagram D' of L then the value of the linking number is the same as for D (regular diagram of L), therefore we shall call this number the linking number of L and denoted by $lk(L)$.

We assuming that the two strands correspond to the two different components because otherwise the move has no affect on linking number. One of the new crossing contributes a $+1$ to the sum, and the other crossing contributes a -1, so the net contribution to the linking number is 0, even if we change the orientation on one of the strands, we will still have one $+1$ and one -1 contribution so the second type move leave the linking number unchanged.

each of the crossings, it is clear that sliding the strands over or under other strands doesn't change the number of $+1's$ and $-1's$ and so the linking number is preserved.

Definition

A knot diagram D is called alternating if, when we proceed along the nodal curve, we pass alternately over, under, over and so on, at each crossing.

need pictures

Definition

Two links L_1 and L_2 are said to be equivalent if there is a homeomorphism of $\mathbf{S^3}$ taking one to the other. Two links L_1 and L_2 are said to be ambient isotopic if there exists a continuous family of homeomorphisms φ_t of $\mathbf{S^3}$ beginning from the identity $\varphi_0 = id$ and ending with a homeomorphism φ_1 with $L_1 = \varphi_1(L_2)$. The ambient isotopy class of a link is called the link type.

Definition

The minimal crossing number $c(K)$ of a given knot type K is the minimum number of crossings among all the planar diagrams representing K.

Definition

A knot invariant is a mathematical object associated with each knot, in such a way that the object attributed to two ambient isotopic knots is the same (or isomorphic in the appropriate category).

For example, the crossing number is a knot invariant. The usual planar diagram for the trefoil knot is alternating. The following result is a well known and very useful characterization of ambient isotopy of knots and links in terms of their planar diagrams.

A seifert surface for a knot K is an orientable surface F such that $\partial F = K$. We will prove that every knot has a seifert surface and from this we can define the genus of a knot. A seifert surface can be also be used to obtain seifert matrices which provide invariant such as the determinate and signature, as well as, relating to other knot invariant such as Alexander Polynomial.

Theorem

(Seifert Algorithm 1934) Given any arbitrary oriented knot (or even link) K, then there exist in $\mathbb{R}^3$ an orientable, connected surface F such that $\partial F = K$.

Remark

1. An orientable connected surface that has as its boundary an orientable knot K is called seifert surface of K.
2. The orientation of F is induced naturally from the orientation of the knot that forms its boundary.
3. The seifert surface depend on the regular diagram D of K *i.e.* the seifert surface formed from D a regular diagram.
4. A seifert surface F of a knot K obtain from the disks and bands as we described before. If we shrink (contract) each disk to a point, and at the same time the width of the bands is shrunk ideally into quite narrow segment, then from these points and segments a graph space is formed such a graph is called seifert graph (of regular diagram D) of K see fig (need picture) is a seifert graph of the Fig8-knot.

Theorem

A closed orientable surface F is homeomorphic to the sphere with several handles (A handel is just a tabular part of the surface) attached to its surface. The number of these handles is called genus of F and denoted by $g(F)$.

Fact: Two connected orientable surfaces are homeomorphic if and only if they have the same number of boundary components and the same genus.

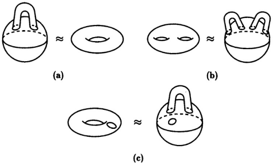
Definition

We call the number of holes in a surface genus of the surface.

For example the sphere has genus 0 and tours has genus 1.

The Calculation of the Genus of the Seifert Surface F of a Knot

Since F has boundary then by above theorem F is homeomorphic to a sphere with several handles attached, and furthermore with a hole on the sphere (with handles) for each components on the link see fig



Fact: An orientable surface with boundary is homeomorphic to one of the closed surface with set of disks removed. It is usually not easy to visualized the seifert surface of a knot. We may consider the minimum genus of all seifert surface for a given knot. This minimum genus is called the genus of K and denoted by $g(K)$. The genus is a knot invariant.

Euler Characteristic

Theorem

We may divide a closed orientable surface F into n_0 points, n_1 edges and n_2 faces.
Let:

$$\chi(F) = n_0 - n_1 + n_2$$

*Then $\chi(F)$ is an integer that is independent of how we divide F i.e. It is only depend on F. This integer is called of **Euler Characteristic** of surface F.*

$$\chi(\mathbf{S}^2) = 2$$

Now the Euler characteristic $\chi(F)$ and the genus of F are related by means of the following equation

$$\chi(F) = 2 - 2g(F)$$

$$g(F) = \frac{2 - \chi(F)}{2}$$

If F has boundary (since the boundary is also composed of several points and edges) the above formula becomes:

$$\chi(F) = 2 - 2g(F) - \mu(F) \tag{0.1}$$

Where $\mu(F)$ s the number of closed curves that is make up the boundary of F.

We may think of the disks and bands of F as a division of F. The points of F in this division are all 4 vortices of each band. The edges of F are the polygonal curve that constitute the edges of the band and the boundaries of the disks between the vertex points, the faces F are the disks and bands.

Let d be the number of disks and b is the number of bands then

$$n_0 = 4b, n_1 = 6b, n_2 = b + d$$

then

$$\chi(F) = 4b - 6b + b + d = d - b$$

Let $\mu(F)$ is the number of components of the link K then from equation (0.1) we obtain

$$d - b = 2 - 2g(F) - \mu(K)$$

or equivalently

$$2g(F)+\mu(K)-1=1-d+b$$

In special case when K is a knot since $\mu(K)=1$ then

$$2g(F)=1-d+b$$

$$g(F)=\frac{1-d+b}{2}$$

Now suppose $\Gamma(D)$ is the seifert graph constructed from seifert surface and since $\Gamma(D)$ is a plane graph is divides $\mathbf{S^2}$ ($\mathbf{S^2}-\{\infty\}\simeq\mathbf{R^2}$) into several domains. In this partition of $\mathbf{S^2}$, the number of points is d and the number of edges is b, suppose that the number of faces is t then

$$2=\chi(\mathbf{S^2})=d-b+t$$

$$t-1=1-d+b$$

i.e. $1 - d + b$ is equal to the number of faces of this division od $\mathbf{S}^2$ excluding the face that contain the point at infinity.

Remark

1. If a knot K has $g(K) = 0$ then it is trivial
2. If a non-trivial knot has a seifert surface of genus 1 then $g(K) = 1$

Suppose that F is a seifert surface created from the regular diagram D, of a knot (or link) K and $\Gamma(D)$ is its seifert graph. We want to create exactly $2g(F)+\mu(K)-1$ closed curves that lie on F when $\Gamma(D)$ partitions $\mathbf{S^2}$, then we showed in previous section that

$$2g(F)+\mu(K)-1=1-d+b=t-1$$

is equal to the number of domains that contain the point ∞. The boundary of each of these domains (faces) is a closed curves, create the closed curves on the seifert surfaces. To calculate the seifert matrix as we will show in the next example the algorithm.

First: Thicken F slightly. create $F \times [0,1]$. The original surface F maybe thought of as $F \times \{0\}$ and so we may say that both α_1 and α_2 lie on $F \times \{0\}$, $\alpha_1 = \alpha_1 \times \{0\}$, $\alpha_2 = \alpha_2 \times \{0\}$ and $\alpha_1^{\#} = \alpha_1 \times \{1\}$, $\alpha_2^{\#} = \alpha_2 \times \{1\}$. We may assign an orientation to α_1 and α_2 in any arbitrary fashions induce in a natural manner orientation on $\alpha_1^{\#}$ and $\alpha_2^{\#}$. We can now calculate the linking number $lk(\alpha_1, \alpha_2^{\#})$ and by the same way we can compute $lk(\alpha_1, \alpha_1^{\#})$, $lk(\alpha_2, \alpha_1^{\#})$ and $lk(\alpha_2, \alpha_2^{\#})$.

These four linking numbers maybe rearranged into the following 2×2 matrix form

$$M = \begin{pmatrix} lk(\alpha_1, \alpha_1^{\#}) & lk(\alpha_1, \alpha_2^{\#}) \\ lk(\alpha_2, \alpha_1^{\#}) & lk(\alpha_2, \alpha_2^{\#}) \end{pmatrix}$$

This matrix is called **Seifert Matrix** of the knot K in the above figure.

If the genus of the seifert surface F of a knot or link is $g(F)$ then on F there are $2g(F)+\alpha\mu(K)-1=m$ closed curves $\alpha_1,\alpha_2,\alpha_3,...\alpha_m$. Expanding the process outlined above, with arbitrary orientation assigned to these closed curves, we can calculate their various linking numbers. As above we may formulate them in terms of the entries of $m\times m$ matrix

$$M=[lk(\alpha_i,\alpha_j^{\#})],i,j=1,2,...,m$$

Notes

1. Since the linking numbers are integers then the matrix M is an integer matrix.
2. The matrix M depends on the orientation of α_1, α_2.
3. the matrix is not an invariant of K.
4. In general the linking numbers $lk(\alpha_i, \alpha_j^{\#})$ and $lk(\alpha_j, \alpha_i^{\#})$ are not equal so that the matrix M is not a symmetric Matrix.
5. If $g(F) = 0$ then the seifert Matrix of K is defined to be the empty matrix (here K is trivial knot))

S-Equivalent of Seifert Matrices

Theorem

Two seifert matrices, obtained from two equivalent knots (or links) can be changed from one to the other by applying a finite number of times the following two operations A_1 and A_2 and their inverse
$A_1 : M_1 \longrightarrow PM_1P^T$ *where P is an invertible integer matrix with* $detP = \pm 1$ *and* P^T *denotes the transpose matrix of P.*

$$A_2 : M_1 \longrightarrow M_2 = \begin{pmatrix} & & & * & 0 & 0 \\ & & & * & 0 & 0 \\ & M_1 & & . & . & . \\ & & & . & . & . \\ 0 & . & . & * & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $*$ denote an arbitrary integers.

Definition

Two square matrices M, M' obtained one from the other by applying the operations A_1, A_2 and the inverse A_2^{-1} a finite number of times are said to be S-Equivalent and are denoted by $M \overset{S}{\sim} M'$

Theorem

Suppose that M and M' are two seifert matrices obtained from seifert surfaces F_1 and F_2 of knot K, then M_1 and M_2 are S-Equivalent.

Suppose that K is an oriented knot or link and $-K$ is the knot with reverse orientation to K then $M_{-K} \overset{S}{\sim} M_K^T$, where M_K^T is the transpose matrix of M_K.

Suppose that K^* is the mirror image of a knot (or link) then $M_{K^*} \overset{S}{\sim} -M_K^T$

If M is the seifert matrix of knot(or link) K then $|\, det(M + M^T)\, |$ is an invariant of the knot K. this invariant is called the determinant of K. We define the determinant of empty matrix to be 1 *i.e.* the determinant of the trivial knot is 1.

Since the determinant of the trefoil is 3, then it is not equivalent to the trivial knot.

Alexander Polynomial: Consider the polynomial

$$det(M - tM^T)$$

The determinant is a polynomial with variable t. We want to show that how this polynomial change when we apply A_1 and A_2, A_2^{-1} Firstly: since $det(P) = det(P^T) = \pm 1$ then

$$det(A(M - tM^T)) = det(P(M - tM^T)P^T) = det(M - tM^T)$$

Therefore, it is not affected by the operation A_1.
Secondly: If we apply A_2 then

$$det(A_2(M - tM^T)) = det \begin{pmatrix} & & & & * & 0 \\ & & & & * & 0 \\ & & M - tM^T & & . & . \\ & & & & . & . \\ & & & & * & 0 \\ - * t & . & . & - * t & 0 & 1 \\ 0 & 0 & 0 & 0 & -t & 0 \end{pmatrix}$$

$$= det \begin{pmatrix} & & & & * & 0 \\ & & & & * & 0 \\ & M - tM^T & & & . & . \\ & & & & . & . \\ & & & & * & 0 \\ 0 & . & . & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -t & 0 \end{pmatrix} = tdet(M - tM^T)$$

imilarly we can obtain $det(A_2^{-1}(M_2 - tM_2^T)) = t^{-1}det(M_1 - tM_1^T)$.

Theorem

Suppose that M_1 and M_2 are the seifert matrices for knot (or link) K. furthermore, if r and s are respectively that orders of M_1 and M_2 then the following equality holds

$$t^{\frac{r}{s}}det(M_1 - tM_1^T) = t^{\frac{s}{r}}det(M_2 - tM_2^T)$$

Therefore, If $M$is a seifert matrix of K and its order is m then

$$t^{-\frac{m}{2}}det(M - tM^T)$$

is invariant of K. This invariant is known as the Alexander polynomial of K and denoted by $\triangle_K(t)$

1. $m = 2g(F) + \mu(K) - 1$
2. $\triangle_K(t)$ has some terms with a negative exponent, however, if we multiply $\triangle_K(t)$ by a suitable factor then we can obtain a polynomial with only positive exponents, sometimes it is preferable to work with such an interpretation of $\triangle_K(t)$.
3. If K is a link with an even number of components then m is odd, therefore for such link $\triangle_K(t)$ is a polynomial with terms as power of $t^{1/2}$ or $t^{-1/2}$.

Theorem

Suppose K is a knot, then $\triangle_K(t)$ is a symmetric laurent polynomial i.e.

$$\triangle_K(t) = a_{-n}t^{-n} + a_{-(n-1)}t^{-(n-1)} + \ldots + a_{n-1}t^{n-1} + a_n t^n$$

and $a_n = a_{-n}$, $a_{n-1} = a_{-(n-1)}, \ldots, a_{-1} = a_1$

$\mid \triangle_K(-1) \mid$ is equal to the determinant of a knot K

Proof.

$\mid \triangle_K(-1) \mid = \mid (-1)^{-m/2} det(M + M^T) \mid = \mid det(M + M^T) \mid$

□

If K is a trivial knot then $\triangle_{\bigcirc}(t) = 1$

If K is the right-hand trefoil knot then $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and rank$= 2$

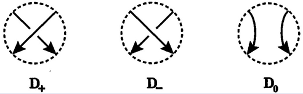
$$\triangle_K(t) = t^{-1} det(M - tM^T) = t^{-1} det \begin{pmatrix} -(1-t) & -t \\ 1 & -(1-t) \end{pmatrix}$$

$$= t^{-1} - 1 + t$$

Definition

Given an oriented knot (or link) K then we may assign to it a laurent polynomial $\nabla_K(z)$, with a fixed variable z by means of the following two axioms

1. If K is the trivial knot, then we assign $\nabla_K(z) = 1$
2. Suppose that $D_+, D_-, D_\circ$ see fig



we need here figure are regular diagrams respectively, of the three knots (or links) $K_+, K_-, K_\circ$
These regular diagrams are exactly the same except at the neighborhood of one crossing point. In this neighborhood that regular diagrams differ in the manner show fig above then the laurent polynomial of the three knots (or links) are related as follows

$$\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_\circ}(z)$$

Note

1. The three regular diagrams $D_+, D_-, D_\circ$ formed as above are called skein diagrams
2. The relation above (**??**), between the laurent polynomials of $K_+, K_-, K_\circ$ is called skein relation.
3. An operation that replace one of $D_+, D_-, D_\circ$ by the other two is called skein operation.
4. The polynomial $\nabla_K(z)$ defined as above is called the Conway Polynomial.

Theorem

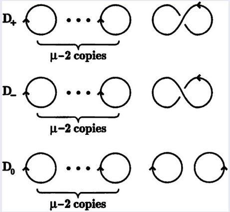
$\triangle_K(t) = \nabla_K(t^{1/2} - t^{-1/2})$

i.e. If we replace z by $t^{1/2} - t^{-1/2}$ in Conway polynomial then the resultant transformation yields the Alexander polynomial. Due to this relationship $\nabla_K(z)$ is often called **Alexander-Conway Polynomial**.

The Alexander Polynomial to the trivial link with μ-components ($\mu \geq 2$) is 0

Proof.

The skein formula corresponding to the skein diagrams see fig



Now since,

$$\nabla_{D_+}(z) - \nabla_{D_-}(z) = z\nabla_{D_\circ}(z)$$

and since both D_+ and D_- are trivial $\mu - 1$-component links then $\nabla_{D_+}(z) = \nabla_{D_-}(z), \Longrightarrow z\nabla_{D_\circ}(z) = 0$ then $\nabla_{D_\circ}(z) = 0$

□

The most affective way to calculate the conway polynomial is to make use of the skein tree. To facilitate our calculation we shall rewrite (**??**) as follows

$$\nabla_{K_+}(z) = \nabla_{K_-}(z) + z\nabla_{K_\circ}(z)$$

(0.2)

$$\nabla_{K_-}(z) = \nabla_{K_+}(z) - z\nabla_{K_\circ}(z) \qquad (0.3)$$

We will take two example to show how we calculate Conway Polynomial.

Suppose that K is the right- hand trefoil knot and D is a regular diagram of K. Step of Calculation

1- Within the dotting circle on D, the crossing point is positive so, we rename this regular diagram by D_+

2- By performing a skein operation, D_+ is transformed into two other regular diagrams. One D_- is the regular diagram obtained by changing the original positive crossing point to negative crossing point. the other $D_\circ$ is obtained by removing the positive crossing point.

3- Connect D_+ and D_- (similarly D_+ and $D_\circ$) by drawing line segment and assign $+1$ (respectively z) to the line segment.

4- Coefficient of $\nabla_{D_-}(z)$ and $\nabla_{D_\circ}(z)$, so for our skein tree diagram we have our first pair of branches, and they correspondence to $1\nabla_{D_-} + z\nabla_{D_\circ}$ in evaluation of $\nabla_{D_+}(z)$.

5- One can see that D_- is equivalent to the trivial knot and hence $\nabla_{D_-}(z) = 1$, therefore D_- will not produce any further branches. $D_\circ$ is not equivalent to the trivial knot or link, and so we can again perform a skein operation within another dotting circle.

6- The crossing point is also positive in the new regular diagram, let us rename $D_\circ$ and D_+, then as before, let us denote the subsequent left hand diagram by D_- and right hand diagram by $D_\circ$. It is easy to see that D_- and $D_\circ$ are respectively the trivial 2-components link and the trivial knot, hence no further branches maybe formed and our skein tree diagram for K is complete Therefore the above example we obtain the following sum

$$\nabla_K(z) = 1\nabla_{\bigcirc}(z) + z\nabla_{\bigcirc\bigcirc}(z) + z^2\nabla_{\bigcirc}(z)$$

Since $\nabla_{\bigcirc}(z) = 1$ and $\nabla_{\bigcirc\bigcirc}(z) = 0$ then the calculation collapses down to

$$\nabla_K(z) = 1 + z^2$$

And since $\nabla_K(z) = \triangle_K(\sqrt{t} - \frac{1}{\sqrt{t}})$ then $\triangle_K(t) = t^{-1} - 1 + t$

The skein tree diagram for the conway polynomial for the fig-8 knot is given in the fig down need picture.
The following calculation is a direct result of this skein tree diagram

$$\nabla_K(z) = \nabla_{\bigcirc}(z) + z\nabla_{\bigcirc\bigcirc}(z) - z^2\nabla_{\bigcirc}(z) = 1 - z^2$$

and

$$\triangle_K(t) = 1 - (\sqrt{t} - \frac{1}{\sqrt{t}})^2 = -t^{-1} + 3 - t$$

If K is a knot then $\triangle_K(1) = 1$

Corollary

If L is a μ-components $(\mu \geq 2)$ link, then $\triangle(L) = 0$

Theorem

Suppose that $f(t)$ is a laurent polynomial that is satisfies the following two conditions

1. $f(1) = 1$
2. $f(t) = f(t^{-1})$

Then there exist a knot that has as its Alexander Polynomial $f(t)$.

Theorem

Suppose that K is a knot then,

1. *If $-K$ is the knot obtained from K by reversing the orientation on K then*

$$\triangle_K(t) = \triangle_{-K}(t)$$

2. *If K^* is the mirror image of K then*

$$\triangle_{K^*}(t) = \triangle_K(t)$$

Theorem

Suppose $K_1 \# K_2$ is a connected sum of two knots (or links) K_1 and K_2 then

$$\triangle_{K_1 \# K_2}(t) = \triangle_{K_1}(t).\triangle_{K_2}(t)$$

Proof.

Firstly, create in the prescribed way the seifert surfaces F_1 and F_2 of, respectively K_1 and K_2. Then the orientable surface formed by joining these surfaces by a band becomes a Seifert surface for $K_1 \# K_2$. If we suppose M_1 and M_2 are the Seifert matrices of K_1 and K_2 obtained from F_1 and F_2 , then M the Seifert matrix of $K_1 \# K_2$ has the following form

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

Therefore

$$det(M - tM^T) = det(M_1 - tM_1^T).det(M_2 - tM_2^T)$$

□

Theorem

Suppose that the genus of a knot K is $g(K)$ then the maximum degree of t in the Alexander Polynomial cannot exceed $g(K)$

A necessary and sufficient condition for the maximum degree of t in $\triangle_K(t)$ the Alexander polynomial of K to be exactly $g(K)$ is $detM \neq 0$

Theorem

Suppose that a link L has a connected incompressible spanning surface of minimal genus. If L can be factorized as $L_1 \# L_2$ then

$$g(L_1 \# L_2) = g(L_1) + g(L_2)$$

Corollary

If $K_1 \# K_2 = \bigcirc$ *then* $K_1 = K_2 = \bigcirc$

Proof.

Since $g(K_1 \# K_2) = g(\bigcirc) = 0$ then $g(K_1) = g(K_2) = 0$, so that K_1 and K_2 are both trivial knots. □

Corollary

If K *is a non-trivial knot with* $g(K) = 1$ *then* K *is a prime.*

Proof.

Suppose K is a composite knot then $K = K_1 \# K_2$ for non-trivial knots K_1, K_2 so that $g(K_1) \geq 1$ and $g(K_2) \geq 1$ then $g(K) = g(K_1) + g(K_2) \geq 2$ and this contradiction therefore K is prime knot. □

We shall define still one more important invariant that depends on the seifert matrix M. The invariant is the signature of a knot. As we noted before that the $\mid det(M + M^T) \mid$ is a knot invariant. However, this is not only knot invariant that can be formed with this symmetric matrix as its core element. but before this we need the next theorem.

Theorem

Suppose A is a $n \times n$ symmetric matrix with its entries real numbers. Then it is possible to find a real (with its entries real numbers) invertible matrix P such that $PAP^T = B$ is a diagonal matrix. In addition, we may assume that $detP = \pm 1$

1. We can diagonalize the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

to

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

and

2. $$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to

$$B = \begin{pmatrix} 2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

Now suppose that the entries in the diagonal of B are $\alpha_1, \alpha_2, \ldots, \alpha_n$ then

1. The number of $\alpha_1, \alpha_2, \ldots, \alpha_n$ that are zero is called the nullity of A, and is denoted by $n(A)$.
2. (The number of positive α_i)-(The number of negative α_i) is called the signature of A, and is denoted by $\sigma(A)$.

1. The signature, $\sigma(A)$ and nullity, $n(A)$ depend only on A, completely irrespective of P chosen
2. The signature may also be obtained from the following $\sigma(A)$=(The number of positive eigne values of A)-(The number of negative eigne vales of A)

Theorem

If N_1 and N_2 are two integer square matrices that are S-equivalent then

$$n(N_1 + N_1^T) = n(N_2 + N_2^T)$$

$$\sigma(N_1 + N_2^T) = \sigma(N_1 + N_2^T)$$

Theorem

Suppose M is a seifert matrix of a knot (or link) K. If we set $A = M + M^T$, then $n(A)$ and $\sigma(A)$ are invariants of the knot (or link) K, Hence we can write $\sigma(A)$ instead of $\sigma(A)$ and this is called the signature of the knot (or link) K, similarly, we can write $n(K)$ instead $n(A)$ and called nullity of K.

Proof.

Since the two matrices M_1 and M_2 of K are S-equivalent, if follows that the signature and nullity of $M_1 + M_1^T$ and $M_2 + M_2^T$ are equal. □

Theorem

If K is a knot, then $n(K) = 0$ and $\sigma(K)$ is always even

Proof.

The seifert matrix M, for K is a square matrix of even order, and since $det(M - M^T) = \triangle_K(1) = 1$, and $det(M + M^T)$ is an odd integer and so non-zero consequently, $n(M + M^T) = 0$, and $n(K) = 0$, therefore the number of eigenvalue of $M + M^T$ that are not zero, hence $\sigma(M + M^T)$ is also even. □

The seifert matrix, M, of the right-hand trefoil knot is

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

Therefore

$$M + M^T = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

We can diagonalize this matrix to obtain

$$\begin{pmatrix} -2 & 0 \\ 0 & -3/2 \end{pmatrix}$$

So, $\sigma(K) = -2$, similarly, the signature of the left-hand trefoil knot is equal 2.

Theorem

The signature of a knot (or link) K has the following properties:

1. $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$
2. *If K^* is the mirror image of K, then $\sigma(K^*) = -\sigma(K)$*
3. *If $-K$ is obtained from K by reversing the orientation on K, then $\sigma(-K) = \sigma(K)$*

If K is achiral knot then $\sigma(K) = 0$

Proof.

If K is achiral knot $(K \simeq K^*)$, consequently $\sigma(K) = \sigma(K^*)$, on the other hand from (2) in the above theorem $\sigma(K^*) = -\sigma(K)$, so $\sigma(K) = \sigma(K^*) = -\sigma(K)$ then $\sigma(K) = 0$

□

If K is a right-hand trefoil knot, we have shown $\sigma(K) = -2$ and this means that K is not achiral.

Theorem

Suppose K is a knot (but not a link) and D is a regular diagram for K, then $\sigma(K)$ can be determined by means of the following three axioms

1. *If K is the trivial knot, then $\sigma(K) = 0$*
2. *If $D_+, D_-, D_\circ$ are the skein diagrams, then*

$$\sigma(D_-) - 2 \leq \sigma(D_+) \leq \sigma(D_-)$$

3. *If $\triangle_K(t)$ is the Alexander Polynomial of K, then $sig(\triangle_K(-1)) = (-1)^{\frac{\sigma(K)}{2}}$*

Notes

1. Because K is a knot (but not a link), D_+ and D_- are regular diagrams of knots, since $D_\circ$ is a link $\sigma(D_\circ)$ cannot be calculated from these regular diagrams
2. We know that $\sigma(K)$ always even, which mean that $\sigma(D_+)$ cannot be $\sigma(D_-) - 1$, therefore, we may rewrite (2) in the theorem above as
$$\sigma(D_+) = \sigma(D_-) \quad \text{or} \quad \sigma(D_+) = \sigma(D_-) - 2 \tag{0.4}$$
3. There are no known similar skein formula that allow us to evaluate the signature of a link.

If we consider the skein diagrams of the figure-8 knot (need picture)
It is easy to see that D_- is the trivial knot so from (0.4)

$$\sigma(D_+) = \sigma(D_-) \quad \text{or} \quad \sigma(D_+) = -2$$

On the other hand, since $\triangle_K(t) = -t^{-1} + 3 - t$ and $\triangle_K(-1) = 5$ so $Sign\triangle_K(-1) = 1$
We may now substitute this positive value into (3) and obtain the equality $1 = (-1)^{\frac{\sigma(K)}{2}}$ then $\sigma(K) = 0$ not -2
We know that figure-8 is achiral so $\sigma(K) = 0$ and this expected result.

Theorem

If $\mu(K)$ is the unknotting number of the knot, then $|\sigma(K)| \leq \mu(K)$

Definition

Suppose K is an oriented knot (or link) and D is a regular (oriented) diagram for K, then the Jones polynomial of K, $J_K(t)$ can be defined uniquely from the following two axioms

Axiom 1 If K is the trivial knot then $J_{\bigcirc}(t) = 1$

Axiom 2 Suppose that D_+, D_- and $D_\circ$ are skein diagrams, then the following skein relation holds

$$1/tJ_{D_+}(t) - tJ_{D_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})J_{D_\circ}(t) \tag{0.5}$$

1. The polynomial itself is a laurent polynomial in $\sqrt{t}$.
2. The polynomial $J_K(t)$ is an invariant of K.
3. The Jones polynomial can be written as the sum of Jones polynomials of the trivial μ-component links $\bigcirc_\mu$ as

$$J_K(t) = f_1 J_{\bigcirc}(t) + f_2 J_{\bigcirc\bigcirc}(t) + \ldots + f_n J_{\bigcirc_\mu}(t)$$

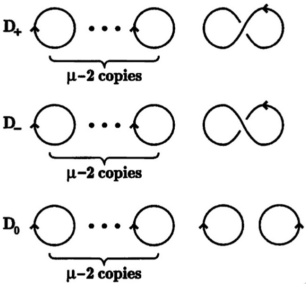
For the trivial μ-component link $\bigcirc_\mu$

$$J_{\bigcirc_\mu}(t) = (-1)^{\mu-1}(\sqrt{t} + \frac{1}{\sqrt{t}})^{\mu-1}$$

The proof will be by induction on μ.
If $\mu = 1$ then this is just Axiom 1, so let us assume for our induction hypothesis that the following holds

$$J_{\bigcirc_{\mu-1}}(t) = (-1)^{\mu-2}(\sqrt{t} + \frac{1}{\sqrt{t}})^{\mu-2}$$

If we now consider the skein diagram



Since $D_+ \cong D_- \cong \bigcirc_{\mu-1}$ and $D_\circ \cong \bigcirc_\mu$ then by the above induction hypothesis and skein relation (0.5) we get

$$1/t(-1)^{\mu-2}(\sqrt{t}+\frac{1}{\sqrt{t}})^{\mu-2}-t(-1)^{\mu-2}(\sqrt{t}+\frac{1}{\sqrt{t}})^{\mu-2}=(\sqrt{t}+\frac{1}{\sqrt{t}})J_{\bigcirc_\mu}(t)$$

since the left hand side of above formula is

$$(-1)^{\mu-2}(\sqrt{t}+\frac{1}{\sqrt{t}})^{\mu-2}(\frac{1}{t}-t)=(-1)^{\mu-1}(\sqrt{t}+\frac{1}{\sqrt{t}})^{\mu-2}(\sqrt{t}+\frac{1}{\sqrt{t}})(\sqrt{t}-\frac{1}{\sqrt{t}})$$

Then the required result follows immediately

The Calculation of Jones Polynomial

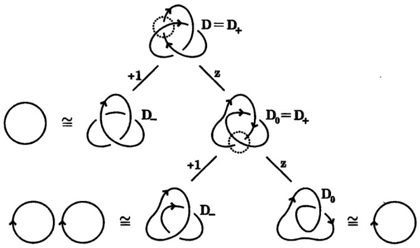
As in the Alexander polynomial case, we can calculate the Jones polynomial by using skein relation after we rewrite it to the following equalities :

$$J_{D_+}(t) = t^2 J_{D_-}(t) + tvJ_{D_\circ}(t) \quad (0.6)$$

$$J_{D_-}(t) = t^{-2} J_{D_+}(t) - t^{-1} vJ_{D_\circ}(t) \quad (0.7)$$

Where $v = (\sqrt{t} - \frac{1}{\sqrt{t}})$

Consider the following figure



from the figure above and by using the equalities (0.6) and (0.7), the Jones polynomial of the right-hand trefoil knot is

$$J_K(t) = t^2 J_{\bigcirc}(t) + t^3 v J_{\bigcirc\bigcirc}(t) + t^2 v^2 J_{\bigcirc} = t + t^3 - t^4$$

Again $v = (\sqrt{t} - \frac{1}{\sqrt{t}})$

Let us denote by $X \bigsqcup Y$ the union of two sets X and Y that have no points in common. for example: we may write the regular diagram of the 2-component trivial link as $\bigcirc \bigsqcup \bigcirc$

$$J_{D \bigsqcup \bigcirc_\mu} = (-1)^\mu (\sqrt{t} + \frac{1}{\sqrt{t}})^\mu J_D(t)$$

Theorem

Suppose that $K_1 \# K_2$ is the connected sum of two knots (or links) then

$$J_{K_1 \# K_2}(t) = J_{K_1}(t).J_{K_2}(t)$$

Theorem

$$J_{D_1 \bigsqcup D_2}(t) = -(\sqrt{t} + \frac{1}{\sqrt{t}})J_{D_1}(t)J_{D_2}(t)$$

Theorem

1. *Suppose K^* is the mirror image of a knot or link, then*

$$J_{K^*}(t) = J_K(t^{-1})$$

2. *Suppose $-K$ is the knot with the reverse orientation to that on K then*

$$J_{-K}(t) = J_K(t)$$